

DERIVED LANGLANDS VIII: LOCAL GL_n

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1. GL_n

Let K/F be a Galois extension of p -adic local fields of characteristic zero. Let \bar{F} denote the algebraic closure of F . In $GL_n K$ let B_K denote the Borel subgroup of upper triangular matrices and let T_K be the torus of diagonal matrices. Let L be the rational maps on T_K and if $\lambda \in L$ we have

$$\lambda : \begin{pmatrix} t_1 & 0 & 0 & \dots & \dots & 0 \\ 0 & t_2 & 0 & \dots & \dots & 0 \\ 0 & 0 & t_3 & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & \dots & t_n \end{pmatrix} \longrightarrow (t_1^m \dots t_n^m)$$

with $m_1, \dots, m_n \in \mathbb{Z}$ we write $\lambda = (m_1, \dots, m_n)$ so $L \cong \mathbb{Z}^n$.

The Lie algebra of GL_n consists of the $n \times n$ matrices and the roots are

$$\begin{aligned} a_1 &= (1, -1, 0, \dots, \dots, 0) \\ a_2 &= (0, 1, -1, 0, \dots, 0) \\ &\vdots \quad \quad \quad \vdots \\ a_n &= (0, 0, \dots, 0, 1, -1). \end{aligned}$$

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Let $(-, -)$ be the familiar bilinear form on the $n \times n$ matrices with entries in K .

2. PREAMBLE

I shall describe how the material of this note constructs a 2-variable L-function for two representations of general linear groups of a p -adic local field F defined over \overline{F} , the algebraic closure of F and admissible in the sense of [60]. More precisely, admissibility in the above sense has the following meaning. We shall say that V is $\mathcal{M}_{cmc,\underline{\phi}}(G)$ -smooth¹ if

$$V = \text{Span}_{(H,\phi) \in \mathcal{M}_{cmc,\underline{\phi}}(G)} V^{(H,\phi)}.$$

In addition we shall say that V is $\mathcal{M}_{cmc,\underline{\phi}}(G)$ -admissible if $\dim_k V^{(H,\phi)} < \infty$ for all $(H, \phi) \in \mathcal{M}_{cmc,\underline{\phi}}(G)$.

I apologise for the state of these notes but I have health problems [47] and impaired vision so this is the best I can do.

We begin with representations ρ and π of $GL_n F$ and $GL_m F$ respectively over \overline{F} which are admissible in the above sense.

We choose an unramified Galois extension of local fields K/F and let $\tilde{\pi}$ be the admissible representation of the semi-direct product $\text{Gal}(K/F) \rtimes GL_m K$ given by base-change [2].

The Galois semi-direct product acts simplicially on the Bruhat-Tits building, for example with the simplicial structure described in [30]. As illustrated in §4, §5, §6 the fundamental subcomplex is a finite number of simplices whose stabiliser is $K^*U_K B_K$ where B_K is a Borel subgroup, U_K is compact open and $U_K B_K$ satisfies the cohomology of §7 and Lemma 9.1.

The observation of Digne-Michel in §4 matches up conjugacy classes in $GL_m F$ with conjugacy classes of elements (Frob_F, A) in the Galois semi-direct product. Therefore the object of the exercise is to construct an L-function $L(s, \rho, \tilde{\pi})$ depending on these conjugacy classes only. A multiplicative Euler characteristic of the form $L(s, \rho, \tilde{\pi})$ may then give a well-defined $L(s, \rho, \pi)$.

This paper explains why $L(s, \rho, \pi)$ in my di- p -adic set up should be given by Artin's usual determinantal formula (see §9 **Part B**).

3. LIE ALGEBRAS AND ROOTS

A generalised Cartan matrix is a square matrix $A = a_{i,j}$ with integer entries in $\sum_n \int D$ where D is diagonal matrices such that (i) each $a_{i,i} = 2$, (ii) if $i \neq j$ then $a_{i,j} \leq 0$ (iii) $a_{i,j} = 0$ if and only if $a_{j,i} = 0$.

¹In previous versions I made a goofy typographical blunder and wrote \cup instead of Span .

For example, for the group G_2 the Cartan matrix is

$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -1 \\ -1 & 2 \end{pmatrix} = DS$$

We can always choose D to have positive diagonal entries. In that case, if S is positive definite then A is said to be a Cartan matrix.

The Cartan matrix of a simple Lie algebra is the matrix whose entries are the scalar products

$$a_{j,i} = 2 \frac{(r_i, r_j)}{(r_j, r_j)},$$

sometimes called the Cartan integers, where the r_i 's are the **simple roots** of the algebra. Condition (i) is obvious. Condition (ii) follows from the fact that if $i \neq j$ then $r_j - 2 \frac{(r_i, r_j)}{(r_j, r_j)} r_i$ is a linear combination of the simple roots r_i and r_j with a positive coefficient for r_j and so the coefficient of r_i has to be non-negative. Condition (iii) follows because orthogonality is a symmetric relation. Finally

$$D_{i,j} = \frac{\delta_{i,j}}{(r_i, r_i)}, \quad S_{i,j} = 2(r_i, r_j).$$

S is positive definite because the simple roots span a Euclidean space.

Conversely, from the Cartan matrix, one can reconstruct the Lie algebra.

Recall that a Lie algebra \mathcal{G} is a vector space with a Lie product $(x, y) \mapsto [x, y]$ which satisfies $[x, y] = -[y, x]$ and the Jacobi identity

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0.$$

Let E be a finite-dimensional Euclidean vector space, with standard inner product $(-, -)$. A root system Φ in E is a set of non-zero vectors (i) spanning E , (ii) The only scalar multiples of $\alpha \in \Phi$ are $\pm\alpha$, (iii) for every root $\alpha \in \Phi$ the set Φ is closed under reflection in the hyperplane perpendicular to α , (iv) if $\alpha, \beta \in \Phi$ then the projection of β onto the line through α is an integer multiple or a half-integer multiple of α .

Equivalently to (iii) and (iv) are the conditions

$$(iii)' \text{ if } \alpha, \beta \in \Phi \text{ then } \sigma_\alpha(\beta) = \beta - 2 \frac{(\alpha, \beta)}{(\alpha, \alpha)} \alpha \in \Phi$$

and

$$(iv)' \text{ if } \alpha, \beta \in \Phi \text{ then } \langle \beta, \alpha \rangle = 2 \frac{(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}.$$

Some authors only include conditions (i)-(iii) in the definition of a root system. In this context a root system which also satisfies (iv) is known as crystallographic and the systems satisfying (ii) are called reduced. Here all root systems will be both crystallographic and reduced.

Note that the pairing $(\alpha, \beta) \mapsto \langle \beta, \alpha \rangle - \Phi \times \Phi \rightarrow \mathbb{Z}$ - is *not* an inner product because it is not symmetric in the variables and is linear only in β .

4. THE OBSERVATION OF DIGNE-MICHEL [26], [27] AND ([59] PP.267-269)

4.1. Let F be a p -adic local field of characteristic zero with algebraic closure \overline{F} and within that a maximal unramified extension inside which I shall assume all our important field elements lie. Let Frob_F be the Frobenius automorphism topologically generating the absolute unramified Galois group of F , $\text{Gal}(F_{\text{ur}}/F)$, so that Frob_F^n generates $\text{Gal}(F_{\text{ur}}/K)$ with $[K : F] = n$.

Assuming that the Lang-Steinberg Theorem holds for U_F which is a subspace of $GL_s K$ acted upon by the unramified Galois group, for any $Y \in GL_s F$ there exists an unramified extension K and a matrix $X \in GL_s K$ such that $Y = X^{-1}\text{Frob}_F(X)$.

Observe that, if $[K : F] = n$,

$$\begin{aligned} X^{-1}\text{Frob}_F(X) &\in GL_s K \\ \iff \text{Frob}_F^n(X^{-1})\text{Frob}_F^{n+1}(X) &= X^{-1}\text{Frob}_F(X) \\ \iff X\text{Frob}_F^n(X^{-1}) &= \text{Frob}_F(X\text{Frob}_F^n(X^{-1})) \\ \iff X\text{Frob}_F^n(X^{-1}) &\in GL_s F. \end{aligned}$$

Let $W \in GL_s$ be another matrix and suppose that

$$(\text{Frob}_F, X^{-1}\text{Frob}_F(X)), (\text{Frob}_F, W^{-1}\text{Frob}_F(W)) \in \text{Gal}(K/F) \times GL_s K$$

are conjugate by $(1, V) \in GL_s K$. Hence, say,

$$\begin{aligned} (1, V)(\text{Frob}_F, X^{-1}\text{Frob}_F(X))(1, V^{-1}) \\ &= (\text{Frob}_F, VX^{-1}\text{Frob}_F(X))(1, V^{-1}) \\ &= (\text{Frob}_F, VX^{-1}\text{Frob}_F(XV^{-1})) \\ &= (\text{Frob}_F, W^{-1}\text{Frob}_F(W)). \end{aligned}$$

Therefore $WVX^{-1} = \text{Frob}_F(WVX^{-1}) \in GL_s F$ and

$$\begin{aligned} (1, WVX^{-1})(\text{Frob}_F^n, X\text{Frob}_F^n(X^{-1}))(1, XV^{-1}W^{-1}) \\ &= (\text{Frob}_F^n, WVX^{-1}X\text{Frob}_F^n(X^{-1}))(1, XV^{-1}W^{-1}) \\ &= (\text{Frob}_F^n, WVX^{-1}X\text{Frob}_F^n(X^{-1}XV^{-1}W^{-1})) \\ &= (\text{Frob}_F^n, WV\text{Frob}_F^n(V^{-1}W^{-1})) \\ &= (\text{Frob}_F^n, WVV^{-1}\text{Frob}_F^n(W^{-1})) \\ &= (\text{Frob}_F^n, W\text{Frob}_F^n(W^{-1})). \end{aligned}$$

Therefore

$(\text{Frob}_F^n, X\text{Frob}_F^n(X^{-1}))$, $(\text{Frob}_F^n, W\text{Frob}_F^n(W^{-1})) \in \text{Gal}(K/F) \times GL_s K$
are conjugate by an element of $GL_s F$ or, equivalently,

$$X\text{Frob}_F^n(X^{-1}), W\text{Frob}_F^n(W^{-1}) \in GL_s F$$

are conjugate in $GL_s F$.

Conversely, if there exists $A \in GL_s F$ such that

$$X\Sigma^n(X^{-1}) = AW\text{Frob}_F^n(W^{-1})A^{-1} \in GL_s F$$

then

$$W^{-1}A^{-1}X = \text{Frob}_F^n(W^{-1}A^{-1}X) \in GL_s K.$$

Also

$$\begin{aligned} & (1, W^{-1}A^{-1}X)(\text{Frob}_F, X^{-1}\text{Frob}_F(X))(1, X^{-1}AW) \\ &= (\text{Frob}_F, W^{-1}A^{-1}XX^{-1}\text{Frob}_F(X))(1, X^{-1}AW) \\ &= (\text{Frob}_F, W^{-1}A^{-1}XX^{-1}\text{Frob}_F(XX^{-1}AW)) \\ &= (\text{Frob}_F, W^{-1}A^{-1}\text{Frob}_F(AW)) \\ &= (\text{Frob}_F, W^{-1}A^{-1}A\text{Frob}_F(W)) \\ &= (\text{Frob}_F, W^{-1}\text{Frob}_F(W)) \end{aligned}$$

so that

$$(\text{Frob}_F, X^{-1}\text{Frob}_F(X)), (\text{Frob}_F, W^{-1}\text{Frob}_F(W)) \in \text{Gal}(K/F) \times GL_s K$$

are conjugate by an element of $GL_s K$.

Therefore we have proved the following result.

Theorem 4.2.

There is a one-one correspondence of the form

$$\left\{ \begin{array}{l} \text{conjugacy} \\ \text{classes in} \\ GL_s F \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} GL_s K - \text{conjugacy} \\ \text{classes of} \\ \text{elements } (\text{Frob}_F, A) \text{ in} \\ \text{Gal}(K/F) \times GL_s K \end{array} \right\}$$

given by

$$X\text{Frob}_F^n(X^{-1}) \leftrightarrow (\text{Frob}_F, X^{-1}\text{Frob}_F(X))$$

for $X \in GL_s M$ for some M/F with $[M : F] = t$.

5. π ON GL_1

In ([41] Lemma 1, p.30), $G = GL_1$, $G_F = F^*$, $G_K = K^*$ we have $K^* = G_K = B_K U_K$, $U_K = \mathcal{O}_K^*$ ². and the vanishing of $H^1(\text{Gal}(K/F); U_K)$ and $H^1(\text{Gal}(K/F); B_K \cap U_K)$ implies that $B_K = K^* = U_K$ and $B_F = F^*$, $U_F = \mathcal{O}_F^*$. The set of roots is empty for GL_1 . $T^K = K^*$, $T_F = F^*$.

$L = \{x \in F^* \mid x \mapsto x^n\} \cong \mathbb{Z}$, rational functions (see [41], p.33). $\hat{L} = \text{Hom}(L, \mathbb{Z}) \cong \mathbb{Z}$, ((see [41], pp.34/35). Also $\hat{M} = \hat{L}^{\text{Gal}(K/F)}$ and $\Lambda(\hat{M})$ is the k -group ring of \hat{M} .

$G_K = B_K = K^*$ and $v = v_F : F^* \rightarrow \mathbb{Z}$ is the normalised valuation on F . $U_K = \mathcal{O}_K^*$ which is compact, as required. $\mathcal{O}_K/\mathcal{P}_K = \mathbb{F}_{q^d}$ with residue degree $[\mathbb{F}_{q^d} : \mathbb{F}_q] = d$. $G_F = B_F = F^*$ and $\mathcal{O}_F/\mathcal{P}_F = \mathbb{F}_q$. We have $\rho : F^* \rightarrow \overline{F}^*$ and $\pi : K^* \rightarrow \overline{F}^*$.

We have $(\mathcal{O}_K^*, \phi)^{K^*}$ so by analogy the the case $\phi = 1$ in [41] and similarly with K replaced by F . We define, for K/F large enough

$$C_c(K^*, (\mathcal{O}_K^*, \phi)^{K^*}) = \{f : K^* \rightarrow \overline{F} \mid f \text{ compactly supported} \\ f(ug) = f(gu) = \phi(u)^{-1}f(g), u \in \mathcal{O}_K^*, g \in K^*\}.$$

and similarly with K replaced by F .

Let v be the normalised F -adic valuation on any extension K/F so if we have $\lambda = (m)$ in $\hat{L} = \text{Hom}(L, \mathbb{Z})$ with $t \in F$ we have $\lambda(t) = t^m$ and if $v(t) = s$ then $t = \pi_F^s u$ with $u \in U_F = \mathcal{O}_F^*$.

We have

$$C_c(K^*, (\mathcal{O}_K^*, \phi)^{K^*}) \rightarrow \overline{F}$$

the details of which will appear later.

² U_F must be compact if $f \neq 0$ and f is compactly supported.

6. π ON GL_2

The Galois action on $K \oplus K$ sends stabilisers of homothety types to themselves. Hence $\text{Gal}(K/F)$ acts on stabilisers of simplices in the building. I believe this is true in general for the action on the Tammo Tom Dieck space of $GL_n K$. The monomial resolution of an admissible representation of GL_2 is given by the double complex of putting the bar resolution of the hyperHecke algebra of the restriction on π to a compact open mod the centre stabiliser of a simplex. The fundamental domain of the GL_2 -action on the building is a 1-simplex which has 2 vertices with compact open modulo the centre stabilisers. The first vertex has stabiliser $G_K = K^* \cdot GL_2 \mathcal{O}_K$ whose elements look like

$$G_K = \left\{ \pi_K^i \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc \in \mathcal{O}_K^*, a, b, c, d \in \mathcal{O}_K, i \in \mathbb{Z} \right\}.$$

We have a Chevalley subgroup U_K as in ([41] Lemma 1, p.30). Inside G_K take the abelian group

$$U_K = \left\{ \begin{pmatrix} \pi_K^i a & 0 \\ \pi_K^i c & \pi_K^i a \end{pmatrix} \mid a \in \mathcal{O}_K^*, c \in \mathcal{O}_K \right\}$$

and the Borel subgroup

$$B_K = \left\{ \begin{pmatrix} \pi_K^j \alpha & \pi_K^j \beta \\ 0 & \pi_K^j \delta \end{pmatrix} \mid \alpha, \delta \in \mathcal{O}_K^*, \beta \in \mathcal{O}_K \right\}$$

so that $G_K = U_K B_K$ and U_K is self-normalising in G_K as the following calculation shows.

$$\begin{aligned} \begin{pmatrix} \pi_K^i a & 0 \\ \pi_K^i c & \pi_K^i a \end{pmatrix} \begin{pmatrix} \pi_K^j \alpha & \pi_K^j \beta \\ 0 & \pi_K^j \delta \end{pmatrix} &= \begin{pmatrix} a\alpha & a\beta \\ c\alpha & c\beta + a\delta \end{pmatrix} \pi_K^{i+j} \\ \begin{pmatrix} \pi_K^j \alpha & \pi_K^j \beta \\ 0 & \pi_K^j \delta \end{pmatrix} \begin{pmatrix} \pi_K^s a_1 & 0 \\ \pi_K^s c_1 & \pi_K^s a_1 \end{pmatrix} &= \begin{pmatrix} \alpha a_1 + \beta c_1 & \beta a_1 \\ \delta c_1 & a_1 \delta \end{pmatrix} \pi_K^{j+s} \end{aligned}$$

so $s = i$ and since $\beta \neq 0$ we have $a = a_1$ which implies $c = 0$, which is generally not true. Also $U_K \cong K^* \times \mathcal{O}_K$ so that $H^1(\text{Gal}(K/F); U_K) = \{1\}$ when K/F is tame and $B_K \cap U_K = K^*$ so that $H^1(\text{Gal}(K/F); B_K \cap U_K) = \{1\}$. This vanishing of cohomology descends to K' in $F \leq K' \leq K$, an intermediate Galois extension.

In the preceding discussion $G_K \leq GL_2K$ stabilises a basic choice of 0-simplex and the conjugate

$$\begin{aligned}
& \begin{pmatrix} \pi_K & 0 \\ 0 & 1 \end{pmatrix} G_K \begin{pmatrix} \pi_K^{-1} & 0 \\ 0 & 1 \end{pmatrix} \\
&= \left\{ \begin{pmatrix} \pi_K & 0 \\ 0 & 1 \end{pmatrix} \pi_K^i \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \pi_K^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} a\pi_K & b\pi_K \\ c & d \end{pmatrix} \begin{pmatrix} \pi_K^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} a & b\pi_K \\ c\pi_K^{-1} & d \end{pmatrix} \right\}.
\end{aligned}$$

which I shall denote by G'_K .

The intersection

$$G''_K = G_K \cap G'_K$$

$$\begin{aligned}
&= K^* \cdot GL_2\mathcal{O}_K \cap \left\{ K^* \cdot \begin{pmatrix} a & b\pi_K \\ c\pi_K^{-1} & d \end{pmatrix} \mid a, b, c, d \in \mathcal{O}_K, ad - bc \in \mathcal{O}_K^* \right\} \\
&= \left\{ \begin{pmatrix} \pi_K^i a & b\pi_K^{i+1} \\ c\pi_K^{i-1} & d\pi_K^i \end{pmatrix} \mid a, b, c, d \in \mathcal{O}_K, ad - bc \in \mathcal{O}_K^*, c \in \pi_K\mathcal{O}_K \right\}
\end{aligned}$$

so we put

$$\begin{aligned}
U''_K &= \left\{ \begin{pmatrix} \pi_K^i a & 0 \\ c\pi_K^{i-1} & a\pi_K^i \end{pmatrix} \mid a, c, d \in \mathcal{O}_K, ad \in \mathcal{O}_K^*, c \in \pi_K\mathcal{O}_K \right\} \\
B''_K &= \left\{ \begin{pmatrix} \pi_K^i a & b\pi_K^{i+1} \\ 0 & d\pi_K^i \end{pmatrix} \mid a, b, d \in \mathcal{O}_K, ad \in \mathcal{O}_K^* \right\}
\end{aligned}$$

The previous discussion shows that U''_K selfnormalises in $G''_K = U''_K \cdot B''_K$ and that the appropriate pair of first Galois cohomology groups both vanish.

Let

$$N_K = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \mid \beta \in \mathcal{O}_K \right\}$$

and set $B_K/N_K = T_K$ and similarly with one or two prime superscripts. We have $\text{Gal}(K/F)$ with $F \leq K$ tame or unramified. Now we want to produce a (unique) conjugacy class in each of $\text{Gal}(K/KF) \rtimes T_K$, $\text{Gal}(K/KF) \rtimes T'_K$ and $\text{Gal}(K/KF) \rtimes T''_K$.

The three compact open modulo the centre subgroups G_K, G'_K and G''_K are the stabilisers of the three simplices comprising the fundamental domain line-segment (see [59] Chapter Two).

7. π ON GL_3

In this example the first vertex has stabiliser $G_K = K^* \cdot GL_3 \mathcal{O}_K$ whose elements look like

$$G_K = \left\{ \pi_K^i X = \pi_K^i \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \mid \det(X) \in \mathcal{O}_K^* \right\}.$$

We have a Chevalley subgroup $U_K \leq G_K$ as in ([41] Lemma 1, p.30) of the form

$$U_K = \left\{ \pi_K^i \begin{pmatrix} a & 0 & 0 \\ d & a & 0 \\ g & h & a \end{pmatrix} \in G_K, i \in \mathbb{Z} \right\}$$

and the Borel subgroup

$$B_K = \left\{ \pi_K^i \begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & k \end{pmatrix} \in G_K, i \in \mathbb{Z} \right\}$$

so that $G_K = U_K B_K$ because

$$\begin{aligned} & \pi_K^i \begin{pmatrix} a & 0 & 0 \\ d & a & 0 \\ g & h & a \end{pmatrix} \pi_K^j \begin{pmatrix} a' & b' & c' \\ 0 & e' & f' \\ 0 & 0 & k' \end{pmatrix} \\ &= \pi_K^{i+j} \begin{pmatrix} aa' & ab' & ac' \\ da' & db' + ae' & dc' + af' \\ ga' & gb' + he' & gc' + hf' + ak' \end{pmatrix}. \end{aligned}$$

In the preceding discussion $G_K \leq GL_2 K$ stabilises a basic choice of 0-simplex and the conjugate

$$\begin{aligned} & \begin{pmatrix} \pi_K & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} G_K \begin{pmatrix} \pi_K^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \left\{ \begin{pmatrix} \pi_K & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \pi_K^i \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \begin{pmatrix} \pi_K^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \\ &= \pi_K^i \begin{pmatrix} a & \pi_K b & \pi_K c \\ \pi_K^{-1} d & e & f \\ \pi_K^{-1} g & h & k \end{pmatrix} \end{aligned}$$

which I shall denote by G'_K and we set $G'''_K = G_K \cap G'_K$, which stabilises one of the 1-simplices of the fundamental 2-simplex.

$$\begin{aligned}
& \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pi_K & 0 \\ 0 & 0 & 1 \end{pmatrix} G_K \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pi_K^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pi_K & 0 \\ 0 & 0 & 1 \end{pmatrix} \pi_K^i \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pi_K^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \\
&= \pi_K^i \begin{pmatrix} a & \pi_K^{-1}b & c \\ \pi_K d & e & \pi_K f \\ g & h\pi_K^{-1} & k \end{pmatrix}
\end{aligned}$$

which gives G''_K and $G^{iv}_K = G_K \cap G''_K$.

$$\begin{aligned}
& \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi_K \end{pmatrix} G_K \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi_K^{-1} \end{pmatrix} \\
&= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi_K \end{pmatrix} \pi_K^i \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi_K^{-1} \end{pmatrix} \right\} \\
&= \pi_K^i \begin{pmatrix} a & b & \pi_K^{-1}c \\ d & e & \pi_K^{-1}f \\ \pi_K g & \pi_K h & k \end{pmatrix}
\end{aligned}$$

which gives G''_K and $G^v_K = G'_K \cap G''_K$. Finally $G^{vi}_K = G_K \cap G' \cap G''_K$. These seven compact open modulo the centre subgroups are the stabilisers of the seven simplices comprising the triangular fundamental domain(see [30] Chapter Seven, flag complexes for GL_n ; particularly p.196 for GL_3). In general

for GL_n of a local field the building has $2^n - 1$ compact modulo the centre stabiliser groups.

8. VANISHING H^1 -CONDITIONS FOR GL_2

Lang's Theorem and H_K for GL_2

I am interested in Lang's Theorem for H_K when K/F is unramified. This is related to but not equivalent to $H^1(\text{Gal}(K/F); H_K) = \{1\}$. I am hoping that if the Lang surjection is onto for G_K, G'_K and G''_K in the GL_2 example (and $H_1 \cap H_2$) that this will give me in a straightforward cohomological way the results [41] on conjugacy classes from the results of §3.

Recall Lang's Theorem. For G over \mathbb{F}_q smooth and $\text{Frob}_{\mathbb{F}_q}$ equalling the Frobenius, temporarily denoted by σ for the rest of this paragraph, then the map $x \mapsto x^{-1}\sigma(x)$ is onto. Here is the proof: Consider $f_a : x \mapsto x^{-1}a\sigma(x)$ whose differential is $(df_a)(1) = d(h \cdot (x^{-1}, a, \sigma(x)))$ where $h(x, y, z) = xyz$. Hence

$$(df_a)(1) = dh_{(1,a,1)} \cdot (-1, 0, d\sigma_1) = -1 + d\sigma_1 = -1.$$

Now $f_a(bx) = f_{f_a(b)}(x)$ so df_b is a bijection for any b . The dimensions of the tangent space and of G at any point are equal so there is a dense open set U containing b and another V containing the image of f_1 . Hence the Lang has a dense open image and is therefore onto.

We know that $H^i(\text{Gal}(K/F); \mathbb{F}_q) = \{1\}$ for all i and , since K/F is unramified, $H^i(\text{Gal}(K/F); \mathcal{O}_F) = \{1\}$ for all i . Therefore, by induction $H^i(\text{Gal}(K/F); \mathcal{P}_F^n) = \{1\}$ for all i and for $n = 0, 1, 2, \dots$

For $n \geq 1$

$$0 \longrightarrow 1 + M_s \mathcal{P}_F^{n+1} \longrightarrow 1 + M_s \mathcal{P}_F^n \longrightarrow M_s \mathbb{F}_q \longrightarrow 0$$

shows that for $n \geq 1$

$$H^i(\text{Gal}(K/F); 1 + M_s \mathcal{P}_F^{n+1}) \xrightarrow{\cong} H^i(\text{Gal}(K/F); 1 + M_s \mathcal{P}_F^n).$$

By the Mittag-Leffler condition, for all i ,

$$H^i(\text{Gal}(K/F); GL_s \mathcal{O}_K)$$

$$\cong \lim_{\leftarrow} H^i(\text{Gal}(K/F); GL_s \mathcal{O}_K / 1 + M_s \mathcal{P}_K^n)$$

$$= \{1\}$$

when K/F is unramified.

Recall how vanishing $H^1(\text{Gal}(K/F); -)$ and finite orders can give Lang's Theorem. Take the Galois module $\overline{\mathbb{F}}_q$ with Frobenius $\sigma \in \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. Take $0 \neq x \in \overline{\mathbb{F}}_q$ so that $x \in \mathbb{F}_{q^d}$ and $x^{q^d-1} = 1$. Set $e = q^d - 1$ and consider $x \in (\mathbb{F}_{q^{de}})^*$. Then the norm $N(x) \in \mathbb{F}_q$ is the product of e copies of $(x\sigma(x)\sigma^2(x)\dots\sigma^{d-1}(x))$ which is trivial because $x^e = 1$ so $N(x) = 1$ and since $H^1(\text{Gal}(\mathbb{F}_{q^{de}}/\mathbb{F}_q); (\mathbb{F}_{q^{de}})^*) = \{1\}$ there exists $y \in (\mathbb{F}_{q^{de}})^*$ such that $y^{-1}\sigma(y) = x$, which is Lang's Theorem.

Since the elements of $GL_s \mathcal{O}_K / 1 + M_s \mathcal{P}_K^n$ have finite order we can prove Lang's Theorem in the same manner for $GL_s \mathcal{O}_K / 1 + M_s \mathcal{P}_K^n$. Then, by Mittag-Leffler, we see that Lang's Theorem holds for $GL_s \mathcal{O}_{F_{unram}}$ where F_{unram}/F is the maximal unramified extension of F .

Once one has Lang surjectivity for $GL_s \mathcal{O}_{F_{unram}}$ the observation of Digne-Michel ([59] p.266-68; see also §3) applies to give a bijection between conjugacy classes in $GL_s \mathcal{O}_{F_{unram}}$ and in $\text{Gal}(F_{unram}/K) \rtimes GL_s \mathcal{O}_{F_{unram}}$.

Here is how we get from Lang's Theorem for $GL_s \mathcal{O}_{F_{unram}}$ to H_K which is the case $F_{unram}^* \cdot GL_s \mathcal{O}_{F_{unram}}$. Suppose we have $\beta^{-1}X$ with $\beta \in F_{unram}^*$ and $X \in GL_s \mathcal{O}_{F_{unram}}$. Therefore we have z such that $z^{-1}\sigma(z) = \beta$ and Y such that $Y^{-1}\sigma(Y) = X$ so that $\sigma(z^{-1})z = \beta^{-1}$. Since the scalar matrices are central we have

$$\begin{aligned} \beta^{-1}X &= \sigma(z^{-1})zY^{-1}\sigma(Y) = Y^{-1}\sigma(Y)\sigma(z^{-1})z \\ &= z^{-1}zY^{-1}\sigma(Y)\sigma(z^{-1})z \\ &= z^{-1}((Yz)^{-1}\sigma(Yz^{-1}))z \\ &= (Yz)^{-1}\sigma(Yz^{-1}) \end{aligned}$$

because z is a scalar matrix.

9. LANGLANDS 2-VARIABLE L-FUNCTIONS ([41] PP. 29-34)

Part A: An ingenious construction ([41] pp. 29-34)

To define the local L -functions, to prove that almost all primes are unramified and to prove that the product over the unramified primes converges for the real part sufficiently large we need some facts from the reduction theory for groups over local fields (see [17]). Much progress has been made in that theory but (at the time of [41]) it was still incomplete. Unfortunately the particular facts needed did not seem then to be in the literature. Very little was lost in just assuming them. For the groups under consideration what is needed is easily verified.

Suppose that K/F is an unramified extension of local fields and G is a quasi-split group over F which splits over K . Let B be a Borel subgroup of G and T a Cartan subgroup of B both of which are defined over F . Let v be the valuation on K . It is a homomorphism from K^* whose kernel is the group of units. \mathcal{O}_K^* . If $t \in T_F$ let $v(t) \in \hat{L}$ be defined by $\langle \lambda, v(t) \rangle = v(\lambda(t))$ for all $\lambda \in L$.

Recall from §2 that $\hat{L} = \text{Hom}(L, \mathbb{Z})$ and from §3 that L is homomorphisms on $T = T^n$ of the form $\lambda \in L$ given by

$$\lambda : \begin{pmatrix} t_1 & 0 & \dots & \dots & 0 \\ 0 & t_2 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & t_n \end{pmatrix} \mapsto t_1^{m_1} t_2^{m_2} \dots t_n^{m_n}$$

with $m_i \in \mathbb{Z}$ we write $\lambda = (m_1, \dots, m_n)$. Thus L is identified with \mathbb{Z}^n .

If $\sigma \in \text{Gal}(K/F)$ then

$$\langle \lambda, \sigma v(t) \rangle = \langle \sigma^{-1} \lambda, v(t) \rangle = v(\sigma^{-1}(\lambda(\sigma t))) = v(\lambda(t))$$

because $\sigma(t) = t$ and $v(\sigma^{-1}a) = v(a)$ for all $a \in K^*$. Thus v is a homomorphism from T_F into \hat{M} , the group of Galois fixed points $\hat{L}^{\text{Gal}(K/F)}$, which is actually a surjection.

Lemma 9.1.

There is a Chevalley lattice in the Lie algebra of G whose stabiliser U_K is invariant under $\text{Gal}(K/F)$. U_K is self-normalising. Moreover, $G_K = B_K U_K$, $H^1(\text{Gal}(K/F); U_K) = \{1\}$ and $H^1(\text{Gal}(K/F); B_K \cap U_K) = \{1\}$. If we choose two such Chevalley lattices with stabilisers U_K and U'_K then U_K and U'_K are conjugate in G_K .

By this we mean that if $gU_K g^{-1} = U'_K$ for $g \in G_F$ then U'_K has the same four properties and $U'_F = G_F \cap U'_K = B_F U'_F$.

Proof of conjugacy of U_F and U'_F in G_F

If $g \in G_K$ and $\sigma \in \text{Gal}(K/F)$, let $g^\sigma = \sigma^{-1}(g)$. If $g \in G_F$ we may write it as $g = bu$ with $b \in B_K$ and $u \in U_K$. Then $g^\sigma = b^\sigma u^\sigma$ and $u^\sigma u^{-1} = b^{-\sigma} b$. By Lemma 9.1 there is $v \in B_K \cap U_K$ such that $u^\sigma u^{-1} = b^{-\sigma} b = v^\sigma v^{-1}$. Then $b' = bv \in B_F$, $u' = v^{-1}u \in U_F = G_F \cap U_K$ and $g = b'u'$ so that $G_F = B_F U_F$.

If $gU_K g^{-1} = U'_K$ for $g \in G_K$ then $g^\sigma U_K g^{-\sigma} = U'_K$ so that $g^{-\sigma} g \in U_K$ which is its own normaliser. By Lemma 9.1 there is $u \in U_K$ so that $g^{-\sigma} g = u^\sigma u^{-1}$. Then $g_1 = gu \in G_F$ and $g_1 U_K g_1^{-1} = U'_K$ so that U_F and U'_F are conjugate in G_F .

Let $C_c(G_F, U_F)$ be the set of all compactly supported functions from G_F to U_F such that $f(gu) = f(ug) = f(g)$ for all $u \in U_F, g \in G_F^3$. $C_c(G_F, U_F)$ is a convolution algebra called the Hecke algebra. If N is the unipotent radical of B let dn be a Haar measure on N_F and let $\frac{d(bnb^{-1})}{dn} = \delta(b)$ if $b \in B_F$. If $\lambda \in \hat{M}$ choose $t \in T_F$ such that $v(t) = \lambda$. If $f \in C_c(G_F, U_F)$ set

$$\hat{f}(\lambda) = \delta^{1/2}(t) \left(\int_{N_F \cap U_F} dn \right)^{-1} \int_{N_F} f(tn) dn.$$

³ U_F must be compact if $f \neq 0$ and f is compactly supported.

The group $\text{Gal}(K/F)$ acts on Ω . Let $\Omega^0 = \Omega^{\text{Gal}(K/F)}$. Ω^0 acts on \hat{M} . Let $\Lambda(M)$ be the complex group algebra of \hat{M} and let $\Lambda^0(\hat{M}) = \Lambda(M)^{\Omega^0}$.

Lemma 9.2. [49]

The map $f \mapsto \hat{f}$ is an isomorphism from $C_c(G_F, U_F)$ to $\Lambda(M)^{\Omega^0}$.

Pages 31 and 32 of [41] assures us that different sets of choices of all this structure leads to canonically compatible data - I shall skip the details.

If π is an irreducible unitary representation of G_F on H whose restriction to U_F contains the identity representation then

$$H_0 = \{x \in H \mid \pi(u)x = x \text{ for all } u \in U_F\}$$

is one-dimensional. If $f \in C_c(G_F, U_F)$ then

$$\pi(f) = \int_{G_F} f(g)\pi(g)dg$$

maps H_0 to itself. The representation of $C_c(G_F, U_F)$ on H_0 determines a ring homomorphism $\chi : C_c(G_F, U_F) \rightarrow \mathbb{C}$ and also $\Lambda^0(\hat{M}) \rightarrow \mathbb{C}$. To define the local L-functions we study such homomorphisms. First of all observe that if χ is associated to a unitary representation then

$$|\chi(f)| \leq \int_{G_F} |f(g)|dg.$$

since $\Lambda(\hat{M})$ is finitely generated over $\Lambda^0(\hat{M})$ any homomorphism like χ can be extended to a ring homomorphism of the form $\Lambda(\hat{M}) \rightarrow \mathbb{C}$. which will necessarily be of the form $\Sigma \hat{f}(\lambda)\lambda \mapsto \Sigma \hat{f}(\lambda)\lambda(t)$ for some $t \in \hat{T}$. Conversely given T the formula above determined a ring homomorphism $\chi_t : \Lambda^0(\hat{M}) \rightarrow \mathbb{C}$. We shall show that $\chi_{t_1} = \chi_{t_2}$ if and only if $t_1 \times \sigma_F$ and $t_2 \times \sigma_F$, where σ_F is the Frobenius substitution, are conjugate in \hat{G}_F . We abbreviate $t \times \sigma$ to $t\sigma$. Every semi-simple conjugacy class of \hat{G}_F is conjugate to some $t\sigma_F$ with $t \in \hat{T}$ [28].

Thus there is a bijection between complex-valued ring homomorphisms of the Hecke algebra and semi-simple conjugacy classes in \hat{G}_F whose projection to $\text{Gal}(K/F)$ is σ_F .

If ρ is a complex analytic representation of \hat{G}_F and $\chi_t : \Lambda^0(\hat{M}) \rightarrow \mathbb{C}$ is the homomorphism associated to π we define the local L-function to be

$$L(s, \rho, \pi) = \frac{1}{\det(1 - \rho(t\sigma_F)|\pi_F|^s)}.$$

Example 9.3. The map $f \mapsto \hat{f}$ is an isomorphism from $C_c(G_F, U_F)$ to $\Lambda^0(M)$. In this example $U_K = K^* = B_K$.

Part B: Left actions

Let K be an unramified extension of F .

In [60] my convention for the structure of the Galois semi-direct product in Lemma 9.1 is $(\sigma, g)(\sigma', g') = (\sigma\sigma', g\sigma(g'))$. The monomial resolution for $\tilde{\pi}$ on

$\text{Gal}(K/F) \rtimes GL_n(K)$, the base change of π , is a sum indexed by stabilisers G_K of the individual simplices in a fundamental domain of $GL_n F$ acting on the Bruhat-Tits building⁴. The function I attach to ρ, π is the Euler product of functions, one for each fundamental simplex stabiliser, which Langlands gives canonically for each piece of π associated to a "special Langlands line".

From the data, monomial resolution of $\tilde{\pi}$, simplex stabiliser, special Langlands line, Langlands gives us, in $\text{Gal}(K/F) \rtimes GL_n(K)$, $(Frob_P, t)$ canonical, with Galois coordinate presumably equal to $Frob_P$. t is an F -point of a torus T dependent on the data. The Galois norm of $(Frob_P, t)$ will be equal to $(Frob_K, t^{[K:F]})$ which contributes

$$\frac{1}{\det(1 - \rho(Frob_K, t^{[K:F]})|\pi_K|^s)},$$

I believe.

Question: What are the analogous Galois norms from the Euler characteristic of the remaining data from the monomial resolution?

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⁴If G_K is a simplex stabiliser for π then $\text{Gal}(K/F) \rtimes G_K$ is a set of simplex stabilisers for the base-change, Galois-permuted in the obvious manner.

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